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Regularity and energy quantization for the Yang–Mills–Dirac equations on 4-manifolds

Takeshi Isobe

Department of Mathematics, Graduate School of Science and Engineering, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

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ABSTRACT

We prove that in 4-dimensional manifolds, any finite energy weak solution to the Yang–Mills–Dirac equations is $W^{2,2} \cap C^0$-gauge equivalent to a C^∞-solution. We also prove energy quantization for a sequence of solutions to the Yang–Mills–Dirac equations in 4-manifolds.

1. Introduction

In this paper, we investigate regularity and energy quantization for solutions to the Yang–Mills–Dirac equations on 4-manifolds. We first introduce some notations and notions in order to introduce our equations.

Let M be an m-dimensional oriented Riemannian manifold. The tangent bundle TM admits an $SO(m)$-structure so that it can be defined by an open covering $\{U_\alpha\}$ and transition maps $g_{\alpha\beta} : U_{\alpha\beta} := U_\alpha \cap U_\beta \to SO(m)$ satisfying the cocycle condition: $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathbf{1}$ in $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, where $\mathbf{1}$ is the identity. Recall that $SO(m)$ is not simply connected. Indeed, $\pi_1(SO(2)) = \mathbb{Z}$ and $\pi_1(SO(m)) = \mathbb{Z}_2$ for $m \geqslant 3$. We mainly concern the case $m = 4$ in this paper and assume $m \geqslant 3$ in the following. Thus there exists a universal covering $\rho : Spin(m) \to SO(m)$. The manifold is said to possess a spin structure if there exist smooth maps $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(m)$ satisfying the cocycle condition $\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} \cdot \tilde{g}_{\gamma\alpha} = \mathbf{1}$ in $U_{\alpha\beta\gamma}$ and $\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}$ for all α, β. A pair (manifold, spin structure) is called a spin manifold. There is a topological obstruction for the existence of a spin structure, namely, the second Stiefel–Whitney class. For this and more, please consult [6,11].

For a spin manifold M, $\{\tilde{g}_{\alpha\beta}\}$ defines a principal $Spin(m)$-bundle which we denote by $P_{Spin}(M)$. It is a double cover of the oriented frame bundle $P_{SO}(M)$ whose restriction to each fiber is $\tilde{\rho} : Spin(m) \to SO(m)$. We can regard $P_{SO}(M)$ as a bundle associated to $P_{Spin}(M)$ via $\rho : Spin(m) \to SO(m)$.

Assume further that m is even. Recall that the Clifford algebra Cl_m is the associative $\mathbb{R}$-algebra with unit, generated by $\mathbb{R}^m$ subject to the relations $uv + vu = -2(u, v)$ for $u, v \in \mathbb{R}^m$ ((u, v) is the standard inner product of u and v in $\mathbb{R}^m$). There exists a complex Cl_m-module $\mathbb{S}_m$ such that $\mathbb{C}l_m := Cl_m \otimes \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}_m)$ as $\mathbb{C}$-algebras. $\mathbb{S}_m$ is the unique (up to isomorphism) irreducible complex Cl_m-module, usually called the spinor module. The orientation on $\mathbb{R}^m$ induces a $\mathbb{Z}_2$-grading on $\mathbb{S}_m$; $\mathbb{S}_m = \mathbb{S}_m^+ \oplus \mathbb{S}_m^-$, see [6,11]. Since $Spin(m) \subset Cl_m^{\mathrm{even}}$, we deduce that each of the spinor spaces $\mathbb{S}_m^\pm$ is a representation space for

E-mail address: isobe.t.ab@m.titech.ac.jp.

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$Spin(m)$. They are in fact irreducible, non-isomorphic complex $Spin(m)$-modules. They are called positive/negative complex spin representations and we denote them as $\rho^{\pm}: Spin(m) \to \mathrm{End}(\mathbb{S}_m^{\pm})$. Associated to these, we obtain Hermitian vector bundles:

$$\mathbb{S}(M) := P_{Spin}(M) \times_{\rho} \mathbb{S}_m,$$
$$\mathbb{S}^{\pm}(M) := P_{Spin}(M) \times_{\rho^{\pm}} \mathbb{S}_m^{\pm}.$$

These are called complex (positive/negative) spinor bundles.

On the other hand, assume that for a compact Lie group G a principal G-bundle $P \to M$ is given. It is defined by an open covering $\{U_\alpha\}$ of M and G-valued transition maps $\{h_{\alpha\beta}\}$ such that the cocycle condition $h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha} = \mathbf{1}$ in $U_{\alpha\beta\gamma}$ is satisfied. Let $\sigma: G \to \mathrm{End}(V)$ be a unitary representation (V a vector space over $\mathbb{C}$). Associated to P, we obtain a vector bundle

$$E = P \times_\sigma V$$

over M with fiber V such that $E|_{U_\alpha} \cong U_\alpha \times V$ and its transition functions are $\{\sigma(h_{\alpha\beta})\}$. For example, if we choose $V = \mathfrak{g}$ the Lie algebra of G and $\sigma =$ the adjoint representation $: G \to \mathrm{Aut}(\mathfrak{g})$, we obtain the adjoint bundle $\mathrm{Ad}(P)$.

Recall also that a smooth connection A on P is defined as $A = \{A_\alpha\}$, where $A_\alpha \in C^\infty(U_\alpha, T^*U_\alpha \otimes \mathfrak{g})$ and satisfies the gluing relation: $A_\beta = h_{\alpha\beta}^{-1} dh_{\alpha\beta} + h_{\alpha\beta}^{-1} A_\beta h_{\alpha\beta}$ in $U_{\alpha\beta}$. The set of all C^∞-connections on P is denoted by $\mathcal{A}^\infty(P)$. For a vector bundle $E = P \times_\sigma V$ and a connection A on P, we have a covariant derivative $\nabla_A^E: C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ which is locally given by $d + \sigma_*(A_\alpha)$, i.e., for a local representation of a section $s = \{s_\alpha\}$ ($s_\alpha \in C^\infty(U_\alpha, V)$), $(\nabla_A^E s)_\alpha := ds_\alpha + \sigma_*(A_\alpha)s_\alpha$, where $\sigma_*: \mathfrak{g} \to \mathrm{End}(V)$ is the derivative of σ. In particular, for $E = \mathrm{Ad}(P)$ we simply write $\nabla_A: C^\infty(M; \mathrm{Ad}(P)) \to C^\infty(M; T^*M \otimes \mathrm{Ad}(P))$ which is locally given by $d + [A_\alpha, \cdot]$.

The covariant derivative ∇_A^E is extended as the covariant exterior derivative $d_A^E: C^\infty(M; \bigwedge^p \otimes E) \to C^\infty(M; \bigwedge^{p+1} T^*M \otimes E)$ which is also defined locally as $d + \sigma_*(A_\alpha)$.

The curvature of a connection A on P is defined as $F_A = (d_A)^2: C^\infty(M; \mathrm{Ad}(P)) \to C^\infty(M; \bigwedge^2 T^*M \otimes \mathrm{Ad}(P))$. It is an algebraic operator, i.e., the multiplication by $\mathfrak{g}$-valued 2-form and we identify it as an element of $C^\infty(M; \bigwedge^2 T^*M \otimes \mathrm{Ad}(P))$; $F_A \in C^\infty(M; \bigwedge^2 T^*M \otimes \mathrm{Ad}(P))$. It is given by $F_A = dA + \frac{1}{2}[A, A]$.

On $\mathbb{S}(M)$ there is a natural connection induced from the Levi-Civita connection on TM, see [6,11]. We also call it the Levi-Civita connection on $\mathbb{S}(M)$. This together with a connection A on P defines a connection $\tilde{\nabla}_A^E$ on the bundle $\mathbb{S}(M) \otimes E$. Namely, it is defined as

$$\tilde{\nabla}_A^E(s \otimes u) = \nabla s \otimes u + s \otimes \nabla_A^E u$$

for $s \in C^\infty(M, \mathbb{S}(M))$ and $u \in C^\infty(M, E)$, where ∇ denotes the Levi-Civita connection on $\mathbb{S}(M)$.

Sections of the vector bundle $\mathbb{S}(M) \otimes E$ are called E-valued spinors. We also call E-valued spinors simply spinors if no confusions arise. The Dirac operator coupled to a connection A acts on E-valued spinors, $\mathfrak{D}_A: C^\infty(M, \mathbb{S}(M) \otimes E) \to C^\infty(M, \mathbb{S}(M) \otimes E)$. It is defined as

$$\mathfrak{D}_A = c \circ \nabla_A : C^\infty\big(M, \mathbb{S}(M) \otimes E\big) \xrightarrow{\nabla_A} C^\infty\big(M, T^*M \otimes \mathbb{S}(M) \otimes E\big) \xrightarrow{c} C^\infty\big(M, \mathbb{S}(M) \otimes E\big),$$

where c denotes the Clifford multiplication $c: T^*M \otimes \mathbb{S}(M) \ni (X, \psi) \mapsto X \cdot \psi \in \mathbb{S}(M)$.

Under these preparations, we now introduce the main object of this paper. For a pair of a connection A on P and an E-valued spinor ψ, we define the action $\mathfrak{A}(A, \psi)$ by the following formula:

$$\mathfrak{A}(A, \psi) = \int_M |F_A|^2 + \langle \psi, \mathfrak{D}_A \psi \rangle \, \mathrm{dvol}_M, \tag{1.1}$$

where the pointwise norm of the curvature F_A is defined from the Riemannian metric on M and a bi-invariant metric on G (which we now fix throughout this paper) and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{S}(M) \otimes E$ defined from the metrics on $\mathbb{S}(M)$ and E.

Critical points of the action $\mathfrak{A}$ on $\mathcal{A}^\infty(P) \times C^\infty(M, \mathbb{S}(M) \otimes E)$ are called Yang–Mills–Dirac fields. They are solutions of the following equations (see [17]):

$$d_A^* F_A = J(\psi), \tag{1.2}$$
$$\mathfrak{D}_A \psi = 0, \tag{1.3}$$

where d_A^* is the L^2-formal adjoint of d_A and

$$J(\psi) = -\frac{1}{2} \langle e_i \cdot \psi, \rho(\sigma_\alpha)\psi \rangle \theta^i \otimes \sigma_\alpha \tag{1.4}$$

is the current. Here $e_1, \ldots, e_m$ is a local orthonormal frame of TM, $\theta^1, \ldots, \theta^m$ the dual coframe of T^*M and $\{\sigma_\alpha\}$ is an orthonormal basis of $\mathfrak{g}$. We have also used and will use the summation convention throughout this paper.

An interesting feature of Eqs. (1.2) and (1.3) is that they are invariant under the action of the gauge group. The gauge group $\mathcal{G}(P)$ of the principal G-bundle P is the set of sections of the bundle $\mathrm{Aut}(P) := P \times_{\mathrm{Ad}} G$, where $\mathrm{Ad} : G \to G$ is the adjoint action $\mathrm{Ad}(g)(h) = ghg^{-1}$. In other words, it is defined as the set of all $h = \{h_\alpha\}$ such that $h_\alpha \in C^\infty(U_\alpha, G)$ and $h_\beta = g_{\alpha\beta}^{-1} h_\alpha g_{\alpha\beta}$ in $U_{\alpha\beta}$. The gauge transformation $h = \{h_\alpha\}$ acts on $\mathcal{A}^\infty(P) \times C^\infty(M, \mathbb{S}(M) \otimes E)$ as $h \cdot (A, \psi) = (h^* A, \sigma(h^{-1})\psi)$, where $h^* A = \{h_\alpha^{-1} dh_\alpha + h_\alpha^{-1} A_\alpha h_\alpha\}$ and $\sigma(h^{-1})\psi = \{\sigma(h_\alpha^{-1})\psi_\alpha\}$ (i.e., h acts on the E-part of ψ). For simplicity, we will write $h \cdot \psi$ for $\sigma(h^{-1})\psi$. The curvature transforms as $F_{h^* A} = h^{-1} F_A h$. It is also easy to see that $\mathfrak{D}_{h^* A}(h \cdot \psi) = h \cdot \mathfrak{D}_A \psi$. Since the metric on G is bi-invariant and the representation σ is unitary, the action $\mathfrak{A}$ is gauge invariant: $\mathfrak{A}(h^* A, h \cdot \psi) = \mathfrak{A}(A, \psi)$.

A special feature in 4-dimensions is that the action $\mathfrak{A}$ is also invariant under conformal transformations of the metric on M. Let the metric on M be denoted by g. For a positive function $\rho \in C^\infty(M)$, a new metric $g_\rho = \rho^2 g$ is defined. The density $|F_A|^2_{g_\rho} \,\mathrm{dvol}_{g_\rho}$ with respect to the metric g_ρ is related to $|F_A|^2_g \,\mathrm{dvol}_g$ as $|F_A|^2_{g_\rho} \,\mathrm{dvol}_{g_\rho} = \rho^{m-4} |F_A|^2_g \,\mathrm{dvol}_g$ (the subscripts specify which metric on M is used). Thus the integral $\int_M |F_A|^2 \,\mathrm{dvol}_M$ is conformally invariant if and only if $m = 4$. On the other hand, the Dirac operator behaves under the conformal transformations of the metric as follows (see [10] and [11]). Let us denote by $\mathfrak{D}_A^\rho$ the Dirac operator as constructed above with respect to the metric g_ρ on M. Then we have the following

$$\mathfrak{D}_A^\rho \psi = \rho^{-\frac{m+1}{2}} \mathfrak{D}_A \big(\rho^{\frac{m-1}{2}} \psi\big).$$

Thus defining $\psi_\rho := \rho^{-\frac{m-1}{2}} \psi$, we have $\langle \psi_\rho, \mathfrak{D}_A^\rho \psi_\rho \rangle \,\mathrm{dvol}_{g_\rho} = \langle \psi, \mathfrak{D}_A \psi \rangle \,\mathrm{dvol}_g$, i.e., the integral $\int_M \langle \psi, \mathfrak{D}_A \psi \rangle \,\mathrm{dvol}_M$ is conformally invariant in any dimension m. Summing up, the action $\mathfrak{A}(A, \psi)$ is conformally invariant if and only if $m = 4$.

Due to these invariance properties, the analysis of Yang–Mills–Dirac equations (YMD for short) becomes difficult and interesting especially in 4-dimensions. It is a kind of critical equations, see [1,8,9,21,31] for critical equations arising in geometry. In this paper, we are mainly interested in the regularity of solutions to YMD and the compactness properties of solution space.

As for the regularity problem, due to the gauge invariance it is not necessary that solutions to the YMD are smooth. For the pure Yang–Mills theory (i.e., $\psi \equiv 0$) in dimension 4, it was proved long ago by Uhlenbeck [29] that a point singularity is removed via a suitable gauge transformation if the Yang–Mills action is finite. Moreover, in [28] Uhlenbeck proved that any $W^{1,2}$-weak solution to the pure Yang–Mills equations in dimension 4 becomes smooth under a $W^{2,2}$-gauge transformation. From the latter result, the removable singularities theorem for pure Yang–Mills fields easily follows (see also Section 2 below).

For the YMD equations, removability of the point singularities is proved by Otway [16] and Li [13]. We first prove in this paper a regularity of finite energy weak solutions (for the definition of energy, see (1.5) below), namely:

Theorem 1.1. *Let $P \to M$ be a principal G-bundle of class $W^{2,2}$ over a 4-dimensional manifold M and $E = P \times_\sigma V$ an associated vector bundle of class $W^{2,2}$. Assume that A and ψ are $W^{1,2}$-connection on P and $L^{8/3}$-section of $\mathbb{S}(M) \otimes E$ respectively, such that (A, ψ) weakly solves YMD on M. Then for any $a \in M$, there exists a ball $B_r(a)$ of radius r with center at a and $g \in W^{2,2}(B_r(a), G) \cap C^0(B_r(a), G)$ such that the gauge transformed field $(g^* A, g \cdot \psi)$ is C^∞ in $B_{r/2}(a)$.*

We shall indicate in Section 2 that the previously known removable singularities theorem follows easily from Theorem 1.1. See [1,8,9,21,31,12] for related regularity problems for critical equations in geometry.

By the works of Uhlenbeck [28], Sedlacek [22], Donaldson [4], Taubes [27] and Rivière [20], the compactness properties of the solution space for the pure Yang–Mills theory (also called moduli space) are now well understood. It is one of important steps to construct 4-manifold invariants (namely, the Donaldson type invariants). Due to the conformal invariance in dimension 4, the moduli space of Yang–Mills equations is not compact in general and energy concentration occurs (also known as bubbling off of instantons), see [4,7]. Namely, assume that $\{A_n\}$ is a sequence of Yang–Mills connections such that the Yang–Mills energy $E_{\mathrm{YM}}(A_n) = E_{\mathrm{YM}}(A_n; M) := \int_M |F_{A_n}|^2 \,\mathrm{dvol}_M$ is uniformly bounded; $\sup_{n \geqslant 1} E_{\mathrm{YM}}(A_n) < +\infty$. Then by the result of Uhlenbeck [28] and Sedlacek [22] (see also [20]), up to suitable gauge transformations there exists a subsequence of $\{A_n\}$ (still denoted by $\{A_n\}$), a finite set $S \subset M$, a function $\theta : S \to \mathbb{R}_{>0}$ and a smooth Yang–Mills connection A on $M \setminus S$ such that $F_{A_n} \rightharpoonup F_A$ weakly in $L^2(M)$ and $|F_{A_n}|^2 \,\mathrm{dvol}_M \to |F_A|^2 \,\mathrm{dvol}_M + \theta \, d\mathcal{H}^0 \lfloor S$ as Radon measures on M, where $\mathcal{H}^0$ is the 0-dimensional Hausdorff measure (i.e., the counting measure). Moreover, under a suitable gauge transformation on $M \setminus S$, A can be extended as a smooth Yang–Mills connection on M, see [29]. The measure $\theta \, d\mathcal{H}^0 \lfloor S$ is called the defect measure for the sequence $\{A_n\}$ and it is proved in [20] that it is quantized, i.e., $\theta(a)$ for each $a \in S$ is a finite sum of energies of Yang–Mills connections on S^4: There exist Yang–Mills connections $B_1, \dots, B_p$ ($p \geqslant 1$) on S^4 such that $\theta(a) = \sum_{i=1}^p E_{\mathrm{YM}}(B_i; S^4)$. This result indicates that there is no loss of energy along the limiting process between bubbles. In [5,18], a similar result is obtained for harmonic maps.

We prove in this paper a similar compactness and quantization property for YMD. As we observed before, since in dimension 4 the YMD is conformally invariant, it is natural to consider a conformally invariant energy functional defined by

$$E(A, \psi) = E(A, \psi; M) = \int_M \big(|F_A|^2 + |\psi|^{\frac{8}{3}}\big) \,\mathrm{dvol}_M \,. \tag{1.5}$$

We also consider the Yang–Mills energy $E_{\mathrm{YM}}(A) = E_{\mathrm{YM}}(A; M) = \int_M |F_A|^2 \,\mathrm{dvol}_M$ and the Dirac energy defined as $E_{\mathrm{Dirac}}(\psi) = E_{\mathrm{Dirac}}(\psi; M) = \int_M |\psi|^{\frac{8}{3}} \,\mathrm{dvol}_M$. Then we have $E(A,\psi) = E_{\mathrm{YM}}(A) + E_{\mathrm{Dirac}}(\psi)$. One can easily check that these energies are conformally invariant: $E(A,\psi_\rho;(M,g_\rho)) = E(A,\psi;(M,g))$, $E_{\mathrm{YM}}(A;(M,g_\rho)) = E_{\mathrm{YM}}(A;(M,g))$ and $E_{\mathrm{Dirac}}(\psi_\rho;(M,g_\rho)) = E_{\mathrm{Dirac}}(\psi;(M,g))$.

Our compactness and energy quantization result is stated as:

Theorem 1.2. *Let $\{(A_n,\psi_n)\}$ be a sequence of solutions to YMD such that $\sup_{n\geqslant 1} E(A_n,\psi_n) < +\infty$. Then there exist a subsequence of $\{(A_n,\psi_n)\}$ (still denoted by $\{(A_n,\psi_n)\}$), principal G-bundles $P_0 \to M$, $P_1 \to S^4, \ldots, P_k \to S^4$ ($k \geqslant 0$) and their associated vector bundles $E_0 = P_0 \times_{\sigma_0} V$, $E_1 = P_1 \times_{\sigma_1} V, \ldots, E_k = P_k \times_{\sigma_k} V$, connections $A_0 \in \mathcal{A}^\infty(P_0)$, $B_1 \in \mathcal{A}^\infty(P_1), \ldots, B_k \in \mathcal{A}^\infty(P_k)$ and spinors $\psi_0 \in C^\infty(M, \mathbb{S}(M)\otimes E_0)$, $\varphi_1 \in C^\infty(M,\mathbb{S}(M)\otimes E_1), \ldots, \varphi_k \in C^\infty(M,\mathbb{S}(M)\otimes E_k)$ such that (A_0,ψ_0) is a solution to the YMD on M, $(B_1,\varphi_1),\ldots,(B_k,\varphi_k)$ are solutions to the YMD on S^4 and the following holds*:

(1) *There exist a finite set S and gauge transformations g_n such that $\{(g_n^* A_n, g_n\cdot\psi_n)\}$ converges to (A_0,ψ_0) in $C^\infty_{\mathrm{loc}}(M\setminus S)$.*
(2) *For some subsequence (still denoted by $\{(A_n,\psi_n)\}$) we have*

$$\lim_{n\to\infty} E_{\mathrm{YM}}(A_n) = E_{\mathrm{YM}}(A_0) + \sum_{i=1}^k E_{\mathrm{YM}}\big(B_i; S^4\big),$$
$$\lim_{n\to\infty} E_{\mathrm{Dirac}}(\psi_n) = E_{\mathrm{Dirac}}(\psi_0) + \sum_{i=1}^k E_{\mathrm{Dirac}}\big(\varphi_i; S^4\big)$$

and

$$\lim_{n\to\infty} E(A_n,\psi_n) = E(A_0,\psi_0) + \sum_{i=1}^k E\big(B_i,\varphi_i; S^4\big).$$

For both the regularity and the compactness-energy quantization results, due to the presence of the spinor fields, one needs to carry out a more subtle analysis than in the case of the pure Yang–Mills. Similar difficulties have been appeared for harmonic maps coupled to spinor fields, namely, Dirac-harmonic maps, see [2], [3] and [33]. Here we overcome these difficulties for the Yang–Mills field coupled to spinors. Our new technical ingredient for such coupled equations is the use of interpolation spaces such as Lorentz spaces and the regularity property of basic geometric operators such as the Laplacian and the Dirac operators acting on these interpolation spaces. The use of interpolation spaces for compactness properties of solutions of critical equations is initiated in [14] and used in [15,20], etc.

2. Regularity of weak solutions

In this section, we prove Theorem 1.1. Our idea of the proof is the local uniqueness of the solution to YMD in various Sobolev spaces.

Fix an arbitrary $a \in M$. We may assume without loss of generality that $E(A,\psi;B(a)) < \epsilon_0$, where ϵ_0 is the Uhlenbeck constant [28,32], $B(a) = B_1(a)$ and the injectivity radius of M is greater than 1. Then by the theorem of Uhlenbeck [28,32], there exists $g \in W^{2,2}(B(a),G)$ such that g^*A is in the Coulomb gauge, i.e., satisfies

$$d^*\big(g^*A\big) = 0 \quad \text{in } B(a), \tag{2.1}$$
$$\iota_\nu\big(g^*A\big) = 0 \quad \text{on } \partial B(a), \tag{2.2}$$

where ν is the outer normal of $\partial B(a)$ and ι_ν is the contraction by ν.

As noted by Taubes [27, Appendix] and Shevchishin [23], such a g has an additional regularity $g \in C^0(B(a),G)$. For simplicity of notation, we may assume that A itself satisfies (2.1) and (2.2). Then from (2.1) and (2.2), we have $\|A\|_{W^{1,2}(B(a))} \leqslant C\epsilon_0$ (see [28,32]).

In $B(a)$, the first equation of YMD (1.2) becomes

$$d^*F_A - *[A, *F_A] = J(\psi).$$

Using $F_A = dA + \frac{1}{2}[A,A]$, it is written as

$$d^*dA + \frac{1}{2}d^*[A,A] - *[A,*dA] - \frac{1}{2}*\big[A,*[A,A]\big] = J(\psi).$$

Thus in the Coulomb gauge (2.1), (2.2), (1.2) becomes

$$\Delta A + \frac{1}{2}d^*[A,A] - *[A,*dA] - \frac{1}{2}*\big[A,*[A,A]\big] = J(\psi), \tag{2.3}$$

where $\Delta = dd^* + d^*d$ is the Hodge Laplacian.

In symbols, we write

$$Q_1(\nabla A, A) = -\frac{1}{2} d^*[A, A] + *[A, *dA]$$

and

$$Q_2(A, A, A) = \frac{1}{2} * \big[A, *[A, A]\big].$$

Then (2.3) is written in the form

$$\Delta A = Q_1(\nabla A, A) + Q_2(A, A, A) + J(\psi). \tag{2.4}$$

We next rewrite the second equation of YMD (1.3). Using an orthonormal frame field $\{e_\alpha\}_{\alpha=1}^4$ on $B(a)$, we have

$$0 = \mathfrak{D}_A \psi = \sum_{\alpha=1}^4 e_\alpha \cdot (\nabla_A)_{e_\alpha} \psi = \sum_{\alpha=1}^4 e_\alpha \cdot \big(\nabla_{e_\alpha} + \sigma_*(A_\alpha)\big)\psi = \mathfrak{D}_0 \psi + \sum_{\alpha=1}^4 \sigma_*(A_\alpha) e_\alpha \cdot \psi, \tag{2.5}$$

where $A_\alpha = \iota(e_\alpha)A$ and $\mathfrak{D}_0 = \sum_{\alpha=1}^4 e_\alpha \cdot \nabla_{e_\alpha}$.

By the Lichnerowicz formula [6,11], we have $\mathfrak{D}_0^2 = \Delta + \frac{s}{4}$ (s is the scalar curvature of the metric on M), and from (2.5) we obtain

$$\begin{aligned}\Delta\psi + \frac{s}{4}\psi = \mathfrak{D}_0^2 \psi &= \sum_{\alpha,\beta=1}^4 e_\beta \cdot \nabla_{e_\beta}\big(-\sigma_*(A_\alpha)e_\alpha \cdot \psi\big)\\ &= -\sum_{\alpha,\beta=1}^4 \big(\nabla_{e_\beta}\big(\sigma_*(A_\alpha)\big)e_\beta \cdot e_\alpha \cdot \psi + \sigma_*(A_\alpha)e_\beta \cdot \nabla_{e_\beta} e_\alpha \cdot \psi + \sigma_*(A_\alpha) e_\beta \cdot e_\alpha \cdot \nabla_{e_\beta}\psi\big).\end{aligned} \tag{2.6}$$

In symbols, we write

$$R_1(\nabla A, \psi) = -\sum_{\alpha,\beta=1}^4 \nabla_{e_\beta}\big(\sigma_*(A_\alpha)\big)e_\beta \cdot e_\alpha \cdot \psi,$$

$$R_2(A, \psi) = -\sum_{\alpha,\beta=1}^4 \sigma_*(A_\alpha) e_\beta \cdot \nabla_{e_\beta} e_\alpha \cdot \psi$$

and

$$R_3(A, \nabla\psi) = -\sum_{\alpha,\beta=1}^4 \sigma_*(A_\alpha) e_\beta \cdot e_\alpha \cdot \nabla_{e_\beta}\psi.$$

Thus (2.6) is written as

$$\Delta\psi = -\frac{s}{4}\psi + R_1(\nabla A, \psi) + R_2(A, \psi) + R_3(A, \nabla\psi). \tag{2.7}$$

For any $\varphi \in C_0^\infty(B(a))$, we have from (2.4) that

$$\begin{aligned}\Delta(\varphi A) &= C_1(\varphi)A + C_2(\varphi)\nabla A + \varphi \Delta A\\ &= C_1(\varphi)A + C_2(\varphi)\nabla A + \varphi\big(Q_1(\nabla A, A) + Q_2(A, A, A) + J(\psi)\big)\\ &= C_1(\varphi)A + C_2(\varphi)\nabla A + C_3(\varphi)Q_3(A, A) + Q_1\big(\nabla(\varphi A), A\big) + Q_2(\varphi A, A, A) + J(\varphi\psi, \psi),\end{aligned} \tag{2.8}$$

where $C_1(\varphi)$, $C_2(\varphi)$ and $C_3(\varphi)$ depend only on φ and the derivatives of φ, $Q_3(A, A)$ is a quadratic form on A and

$$J\big(\psi, \psi'\big) = -\frac{1}{2}\langle e_i \cdot \psi, \rho(\sigma_\alpha)\psi'\rangle \theta^i \otimes \sigma_\alpha.$$

On the other hand, we have from (2.7) that

$$\begin{aligned}\Delta(\varphi\psi) &= d_1(\varphi)\psi + d_2(\varphi)\nabla\psi + \varphi\Delta\psi\\ &= d_1(\varphi)\psi + d_2(\varphi)\nabla\psi + \varphi\left(-\frac{s}{4}\psi + R_1(\nabla A, \psi) + R_2(A, \psi) + R_3(A, \nabla\psi)\right)\\ &= d_4(\varphi)\psi + d_2(\varphi)\nabla\psi + d_3(\varphi)R_2(A, \psi) + R_1\big(\nabla(\varphi A), \psi\big) + R_3\big(A, \nabla(\varphi\psi)\big),\end{aligned} \tag{2.9}$$

where $d_1(\varphi)$, $d_2(\varphi)$, $d_3(\varphi)$ and $d_4(\varphi)$ depend only on s, φ and the derivatives of φ and $R_2(A,\psi)$ here is different from that was previously defined above, but it is a bilinear form on A and ψ as before.

Defining

$$\mathfrak{Q}\left(A,\psi,\begin{pmatrix} B\\ \eta\end{pmatrix}\right)=\begin{pmatrix} Q_1(\nabla B,A)+Q_2(B,A,A)+J(\eta,\psi)\\ R_1(\nabla B,\psi)+R_3(A,\nabla\eta)\end{pmatrix}$$

for $(B,\eta)\in W^{1,2}(B(a),T^*B(a)\otimes\mathfrak{g})\times L^{8/3}(B(a),\mathbb{S}_m\otimes V)$, Eqs. (2.8) and (2.9) are written as

$$\Delta\begin{pmatrix}\varphi A\\ \varphi\psi\end{pmatrix}-\mathfrak{Q}\left(A,\psi,\begin{pmatrix}\varphi A\\ \varphi\psi\end{pmatrix}\right)=\begin{pmatrix} C_1(\varphi)A+C_2(\varphi)\nabla A+C_3(\varphi)Q_3(A,A)\\ d_4(\varphi)\psi+d_2(\varphi)\nabla\psi+d_3(\varphi)R_2(A,\psi)\end{pmatrix}. \tag{2.10}$$

Let us denote the closure of $C_0^\infty(B(a))$ in $W^{1,p}(B(a))$ by $W_0^{1,p}(B(a))$ and set $W_0^{k,p}(B(a))=W^{k,p}(B(a))\cap W_0^{1,p}(B(a))$. We have the following lemma:

Lemma 2.1. *Let us assume $\mu>0$. There exists $0<\epsilon<\epsilon_0$ such that for $A\in W^{1,2}(B(a),T^*B(a)\otimes\mathfrak{g})$ satisfying (2.1), (2.2) and $\psi\in L^{8/3}(B(a),\mathbb{S}_m\otimes V)$ with $E(A,\psi;B(a))<\epsilon$, the operator*

$$\Delta+\mu-\mathfrak{Q}\left(A,\psi,\begin{pmatrix}\cdot\\ \cdot\end{pmatrix}\right): W_0^{1,2}\big(B(a),T^*B(a)\otimes\mathfrak{g}\big)\oplus W_0^{1,8/5}\big(B(a),\mathbb{S}_m\otimes V\big)$$
$$\to W^{-1,2}\big(B(a),T^*B(a)\otimes\mathfrak{g}\big)\oplus W^{-1,8/5}\big(B(a),\mathbb{S}_m\otimes V\big)$$

is invertible.

Proof. Let us assume $\tilde{A}\in W_0^{1,2}(B(a),T^*B(a)\otimes\mathfrak{g})$ and $\tilde{\psi}\in W_0^{1,8/5}(B(a),\mathbb{S}_m\otimes V)$. For $B\in W_0^{1,2}(B(a),T^*B(a)\otimes\mathfrak{g})$, we have by the Hölder's inequality and the Sobolev embedding $W^{1,2}\subset L^4$ in dimension 4 that

$$\left|\left\langle Q_1(\nabla\tilde{A},A),B\right\rangle\right|\leqslant C\|A\|_4\|\nabla\tilde{A}\|_2\|B\|_4\leqslant C\|A\|_4\|\tilde{A}\|_{1,2}\|B\|_{1,2}, \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $W^{-1,2}$ and $W^{1,2}$, $\|\cdot\|_p$ is the L^p-norm on $B(a)$ and $\|\cdot\|_{k,p}$ is the $W^{k,p}$-norm on $B(a)$.

From (2.11), we have

$$\left\|Q_1(\nabla\tilde{A},A)\right\|_{-1,2}\leqslant C\|A\|_4\|\tilde{A}\|_{1,2}. \tag{2.12}$$

Similarly, we have

$$\left\|Q_2(\tilde{A},A,A)\right\|_{-1,2}\leqslant C\|\tilde{A}\|_{1,2}\|A\|_4^2 \tag{2.13}$$

and

$$\left\|J(\tilde{\psi},\psi)\right\|_{-1,2}\leqslant C\|\tilde{\psi}\|_{1,8/5}\|\psi\|_{8/3}. \tag{2.14}$$

On the other hand, for $\eta\in W^{1,8/3}(B(a),\mathbb{S}_m\otimes V)$ we have by the Hölder's inequality and the Sobolev embedding $W^{1,8/3}\subset L^8$ in dimension 4 that

$$\left|\left\langle R_1(\nabla\tilde{A},\psi),\eta\right\rangle\right|\leqslant C\|\nabla\tilde{A}\|_2\|\psi\|_{8/3}\|\eta\|_8\leqslant C\|\tilde{A}\|_{1,2}\|\psi\|_{8/3}\|\eta\|_{1,8/3}. \tag{2.15}$$

Thus we have

$$\left\|R_1(\nabla\tilde{A},\psi)\right\|_{-1,8/5}\leqslant C\|\tilde{A}\|_{1,2}\|\psi\|_{8/3}. \tag{2.16}$$

Similarly we have

$$\left\|R_3(A,\nabla\tilde{\psi})\right\|_{-1,8/5}\leqslant C\|A\|_{1,2}\|\tilde{\psi}\|_{1,8/5}. \tag{2.17}$$

From (2.12), (2.13), (2.14), (2.16) and (2.17) we have

$$\left\|\mathfrak{Q}\left(A,\psi,\begin{pmatrix}\tilde{A}\\ \tilde{\psi}\end{pmatrix}\right)\right\|_{W^{-1,2}(B(a))\oplus W^{-1,8/5}(B(a))}\leqslant C\epsilon\left\|\begin{pmatrix}\tilde{A}\\ \tilde{\psi}\end{pmatrix}\right\|_{W_0^{1,2}(B(a))\oplus W_0^{1,8/5}(B(a))}. \tag{2.18}$$

Since $\Delta+\mu: W_0^{1,2}(B(a))\oplus W_0^{1,8/5}(B(a))\to W^{-1,2}(B(a))\oplus W^{-1,8/5}(B(a))$ is invertible, the assertion follows from (2.18) if $\epsilon>0$ is small. $\square$

We next prove the following:

Lemma 2.2. *Let $\mu > 0$ and $1 < p < 2$ be arbitrary. There exists $0 < \epsilon < \epsilon_0$ such that for $A \in W^{1,2}(B(a), T^*B(a) \otimes \mathfrak{g})$ satisfying (2.1), (2.2) and $\psi \in L^{8/3}(B(a), \mathbb{S}_m \otimes V)$ with $E(A, \psi; B(a)) < \epsilon$, the operator*

$$\Delta + \mu - \mathfrak{Q}\left(A, \psi, \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}\right) : W_0^{2,p}\big(B(a), T^*B(a) \otimes \mathfrak{g}\big) \oplus W_0^{2,\frac{8p}{8+p}}\big(B(a), \mathbb{S}_m \otimes V\big)$$
$$\to L^p\big(B(a), T^*B(a) \otimes \mathfrak{g}\big) \oplus L^{\frac{8p}{8+p}}\big(B(a), \mathbb{S}_m \otimes V\big)$$

is invertible.

Proof. For $1 < p < 2$, define $q = \frac{p}{p-1}$. For $\tilde{A} \in W_0^{2,p}(B(a), T^*B(a) \otimes \mathfrak{g})$ and $B \in L^q(B(a), T^*B(a) \otimes \mathfrak{g})$ we have by the Hölder's inequality and the Sobolev embedding $W^{2,p} \subset L^{\frac{4p}{4-p}}$ in dimension 4 that

$$\big|\langle Q_1(\nabla\tilde{A}, A), B\rangle\big| \leqslant C\|A\|_4 \|\nabla\tilde{A}\|_{\frac{4p}{4-p}} \|B\|_q \leqslant C\|A\|_{1,2}\|\tilde{A}\|_{2,p}\|B\|_q, \tag{2.19}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between L^p and L^q.

Thus we have

$$\big\|Q_1(\nabla\tilde{A}, A)\big\|_p \leqslant C\|A\|_{1,2}\|\tilde{A}\|_{2,p}. \tag{2.20}$$

Similarly, we have

$$\big\|Q_2(\tilde{A}, A, A)\big\|_p \leqslant C\|A\|_{1,2}^2\|\tilde{A}\|_{2,p}. \tag{2.21}$$

On the other hand, for $\tilde{\psi} \in W_0^{2,\frac{8p}{8+p}}(B(a), \mathbb{S}_m \otimes V)$ we have $\tilde{\psi} \in L^{\frac{8p}{8-3p}}(B(a), \mathbb{S}_m \otimes V)$ by the Sobolev embedding and

$$\big\|J(\tilde{\psi}, \psi)\big\|_p \leqslant C\|\tilde{\psi}\|_{\frac{8p}{8-3p}}\|\psi\|_{8/3} \leqslant C\|\tilde{\psi}\|_{2,\frac{8p}{8+p}}\|\psi\|_{8/3} \tag{2.22}$$

by the Hölder's inequality.

As for $R_1(\nabla\tilde{A}, \psi)$, since $\nabla\tilde{A} \in W^{1,p} \subset L^{\frac{4p}{4-p}}$ and $\psi \in L^{8/3}$, we have by the Hölder's inequality

$$\big\|R_1(\nabla\tilde{A}, \psi)\big\|_{\frac{8p}{p+8}} \leqslant C\|\nabla\tilde{A}\|_{\frac{4p}{4-p}}\|\psi\|_{8/3} \leqslant C\|\tilde{A}\|_{2,p}\|\psi\|_{8/3}. \tag{2.23}$$

Similarly, since $\nabla\tilde{\psi} \in W^{1,\frac{8p}{p+8}} \subset L^{\frac{8p}{8-p}}$ we have

$$\big\|R_3(A, \nabla\tilde{\psi})\big\|_{\frac{8p}{8+p}} \leqslant C\|\nabla\tilde{\psi}\|_{\frac{8p}{8-p}}\|A\|_4 \leqslant C\|\tilde{\psi}\|_{2,\frac{8p}{8+p}}\|A\|_4. \tag{2.24}$$

From (2.20)–(2.24), we have

$$\left\|\mathfrak{Q}\left(A, \psi, \begin{pmatrix} \tilde{A} \\ \tilde{\psi} \end{pmatrix}\right)\right\|_{L^p \oplus L^{\frac{8p}{8+p}}} \leqslant C\epsilon \left\|\begin{pmatrix} \tilde{A} \\ \tilde{\psi} \end{pmatrix}\right\|_{W_0^{2,p} \oplus W_0^{2,\frac{8p}{8+p}}}. \tag{2.25}$$

Since $\Delta + \mu : W_0^{2,p} \oplus W_0^{2,\frac{8p}{8+p}} \to L^p \oplus L^{\frac{8p}{8+p}}$ is invertible, the assertion follows from (2.25). $\square$

Under these preparations, we now give the proof of Theorem 1.1.

Proof of Theorem 1.1. Recall that at the beginning of this section, we have assumed that A is in the Coulomb gauge, i.e., it satisfies (2.1) and (2.2). In particular, $A \in W^{1,2}(B(a))$. On the other hand, for arbitrary $\phi \in C_0^\infty(B(a))$, since $(A, \psi) \in W^{1,2}(B(a), T^*B(a) \otimes \mathfrak{g}) \oplus L^{8/3}(B(a), \mathbb{S}_m \otimes V)$ is a solution to YMD, we have from (2.5) and the Hölder's inequality that

$$\mathfrak{D}_0(\phi\psi) = \phi\mathfrak{D}_0\psi + \nabla\phi \cdot \psi = -\phi\sum_{\alpha=1}^{4}\sigma_*(A_\alpha)e_\alpha \cdot \psi + \nabla\phi \cdot \psi \in L^{8/5}\big(B(a)\big). \tag{2.26}$$

Therefore by the elliptic regularity we have $\phi\psi \in W^{1,8/5}(B(a))$ and $\psi \in W^{1,8/5}_{\mathrm{loc}}(B(a))$.

Let $\varphi \in C_0^\infty(B(a))$ be arbitrary. By Lemma 2.2 and the observation we have made, for any $1 < p < 2$ there exists a unique $(\tilde{A}, \tilde{\psi}) \in W_0^{2,p}(B(a), T^*B(a) \otimes \mathfrak{g}) \oplus W_0^{2,\frac{8p}{8+p}}(B(a), \mathbb{S}_m \otimes V)$ satisfying

$$\Delta\begin{pmatrix} \tilde{A} \\ \tilde{\psi} \end{pmatrix} + \mu\begin{pmatrix} \tilde{A} \\ \tilde{\psi} \end{pmatrix} - \mathfrak{Q}\left(A, \psi, \begin{pmatrix} \tilde{A} \\ \tilde{\psi} \end{pmatrix}\right) = \begin{pmatrix} C_1(\varphi)A + C_2(\varphi)\nabla A + C_3(\varphi)Q_3(A, A) \\ d_4(\varphi)\psi + d_2(\varphi)\nabla\psi + d_3(\varphi)R_2(A, \psi) \end{pmatrix} + \mu\begin{pmatrix} \varphi A \\ \varphi\psi \end{pmatrix}. \tag{2.27}$$

Since $(\varphi A, \varphi\psi) \in W_0^{1,2}(B(a), T^*B(a)\otimes\mathfrak{g}) \oplus W_0^{1,8/5}(B(a), \mathbb{S}_m \otimes V)$ satisfies (2.27) for $(\tilde{A}, \tilde{\psi})$ replaced by $(\varphi A, \varphi\psi)$ and $(\tilde{A}, \tilde{\psi})$ is also a solution in the class $W_0^{1,2} \oplus W_0^{1,8/5}$ if $4/3 \leqslant p < 2$, we have by the uniqueness of the solution in that class (Lemma 2.1) that $(\tilde{A}, \tilde{\psi}) = (\varphi A, \varphi\psi)$. Thus we have $(\varphi A, \varphi\psi) \in W_0^{2,p}(B(a), T^*B(a)\otimes\mathfrak{g}) \oplus W_0^{2,\frac{8p}{8+p}}(B(a), \mathbb{S}_m \otimes V)$ for $4/3 \leqslant p < 2$. From this and the Sobolev embedding, we have for any $B' \Subset B(a)$, any $p_1 < \infty$, $p_2 < 4$, $q_1 < 8/3$ and $q_2 < 8$ that (choosing the cutoff function φ suitably and $\epsilon > 0$ small if necessary)

$$A \in L^{p_1}(B'), \tag{2.28}$$
$$\nabla A \in L^{p_2}(B'), \tag{2.29}$$
$$\nabla\psi \in L^{q_1}(B'), \tag{2.30}$$
$$\psi \in L^{q_2}(B'). \tag{2.31}$$

Then we choose a cutoff function φ such that supp(φ) is slightly smaller than B' and we write Eq. (2.27) for $(\tilde{A}, \tilde{\psi}) = (\varphi A, \varphi\psi)$ as

$$\Delta\begin{pmatrix}\varphi A\\ \varphi\psi\end{pmatrix} + \mu\begin{pmatrix}\varphi A\\ \varphi\psi\end{pmatrix} = \mathfrak{Q}\left(A, \psi, \begin{pmatrix}\varphi A\\ \varphi\psi\end{pmatrix}\right) + \begin{pmatrix}C_1(\varphi)A + C_2(\varphi)\nabla A + C_3(\varphi)Q_3(A, A)\\ d_4(\varphi)\psi + d_2(\varphi)\nabla\psi + d_3(\varphi)R_2(A, \psi)\end{pmatrix} + \mu\begin{pmatrix}\varphi A\\ \varphi\psi\end{pmatrix}. \tag{2.32}$$

Then for any $p_3 < 4$ and $q_3 < 8/3$, choosing p_1, p_2, q_1 and q_2 suitably (and hence choosing $\epsilon > 0$ small if necessary), we have

$$\text{the first component of } \mathfrak{Q} \text{ is in } L^{p_3}(B(a)), \tag{2.33}$$
$$\text{the second component of } \mathfrak{Q} \text{ is in } L^{q_3}(B(a)), \tag{2.34}$$
$$C_1(\varphi)A + C_2(\varphi)\nabla A + C_3(\varphi)Q(A, A) + \mu\varphi A \in L^{p_3}(B(a)), \tag{2.35}$$
$$d_1(\varphi)\psi + d_2(\varphi)\nabla\psi + d_3(\varphi)R_2(A, \psi) + \mu\varphi\psi \in L^{q_3}(B(a)). \tag{2.36}$$

From (2.33)–(2.36), we have by the elliptic regularity that

$$\varphi A \in W^{2,p_3}(B(a)), \tag{2.37}$$
$$\varphi\psi \in W^{2,q_3}(B(a)). \tag{2.38}$$

Thus for any $B'' \Subset B'$ and any $p_4 < \infty$ and $q_4 < 8$, we obtain by the Sobolev embedding theorem (and choosing the cutoff function φ and p_i, q_i $(1 \leqslant i \leqslant 3)$ suitably) that

$$A \in C^0(B''), \tag{2.39}$$
$$\nabla A \in L^{p_4}(B''), \tag{2.40}$$
$$\psi \in C^0(B''), \tag{2.41}$$
$$\nabla\psi \in L^{q_4}(B''). \tag{2.42}$$

We repeat the similar argument: First choose φ such that supp$(\varphi) \subset B''$. For any $p_5 < \infty$ and $q_5 < 8$, choosing p_4 and q_4 suitably (and hence choosing $\epsilon > 0$ small if necessary), we have from Eq. (2.32) for this choice of φ that

$$\text{the first component of } \mathfrak{Q} \text{ is in } L^{p_5}(B(a)), \tag{2.43}$$
$$\text{the second component of } \mathfrak{Q} \text{ is in } L^{q_5}(B(a)), \tag{2.44}$$
$$C_1(\varphi)A + C_2(\varphi)\nabla A + C_3(\varphi)Q(A, A) + \mu\varphi A \in L^{p_5}(B(a)), \tag{2.45}$$
$$d_1(\varphi)\psi + d_2(\varphi)\nabla\psi + d_3(\varphi)R_2(A, \psi) + \mu\varphi\psi \in L^{q_5}(B(a)). \tag{2.46}$$

(2.43)–(2.46) imply that

$$A \in W^{2,p_5}(B(a)), \tag{2.47}$$
$$\psi \in W^{2,q_5}(B(a)). \tag{2.48}$$

By the Sobolev embedding, we obtain from (2.47) and (2.48) that

$$A \in C^1(B''), \tag{2.49}$$
$$\psi \in C^1(B''). \tag{2.50}$$

Once (2.49) and (2.50) are proved, from (2.4) and (2.5) we obtain $A \in C^\infty(B'')$ and $\psi \in C^\infty(B'')$. This completes the proof of Theorem 1.1. □

As a corollary of the proof of Theorem 1.1, we have the following bound of (A,ψ) under the small energy hypothesis.

Corollary 2.1. *For any integer $k \geqslant 0$, there exist $\epsilon > 0$ and $C = C(k,\epsilon) > 0$ such that if $E(A,\psi; B(a)) < \epsilon$ and (A,ψ) is a solution of YMD in the Coulomb gauge (2.1), (2.2), we have*

$$\|A\|_{C^k(B_{1/2}(a))} + \|\psi\|_{C^k(B_{1/2}(a))} \leqslant C(k,\epsilon).$$

Before ending this section, we give a remark about the removable singularities theorem for YMD. We sketch the argument which shows that it easily follows from Theorem 1.1 and the Uhlenbeck's removable singularities theorem for Sobolev bundles [30].

For finitely many points $a_1,\dots,a_p \in M$, set $M' = M \setminus \{a_1,\dots,a_p\}$. Let $P' \to M'$ be a smooth G-bundle and (A,ψ) a solution to YMD with finite energy $E(A,\psi;M') < \infty$. Thus, in particular, $\int_{M'} |F_A|^2 \,\mathrm{dvol}_M < \infty$. By the theorem of Uhlenbeck [30], $P' \to M'$ can be extended as a C^∞-bundle over M, i.e., there exists a C^∞-bundle $P \to M$ such that $P|_{M'} \cong P'|_{M'}$. Moreover, A can be extended as a $W^{1,2}$-connection on P. This extension is obtained by showing that for some neighborhood U_i of a_i, there exists $g_i \in W^{2,2}_{\mathrm{loc}}(U_i \setminus \{a_i\}, G)$ such that g_i^*A is in $W^{1,2}(U_i)$. By approximating g_i, we may assume that g_i is smooth in $U_i \setminus \{a_i\}$. Then by setting $A_i = g_i^*A$ in U_i, $A|_{M'}$ and $\{A_i\}_{i=1}^p$ define a $W^{1,2}$-connection A' on P. Using g_i, ψ is also extended to M by setting $\psi' = \psi$ in M' and $\psi' = \sigma(g_i^{-1})\psi$ in U_i. Then it is obvious $\psi' \in L^{8/3}(M)$. The standard cutoff argument shows that (A',ψ') is a finite energy weak solution to YMD on M. Now by Theorem 1.1, there exists $g \in W^{2,2}(M,\mathrm{Aut}(P))$ such that $(g^*A, g\cdot\psi)$ is a smooth solution to YMD. In fact, since the gauge is in Coulomb, it is in the class C^0 (see [27,23]). Thus we recover the removable singularities theorem [16,13] from Theorem 1.1.

3. Energy gap

Throughout this section, we assume $M = S^4$, the standard 4-sphere. To ensure that the number of bubbles in Theorem 1.2 is finite, we need the following energy gap theorem (see [4,17] for related results).

Proposition 3.1. *There exists $\epsilon_0 > 0$ such that if (A,ψ) is a solution to YMD on S^4 and $E(A,\psi;S^4) < \epsilon_0$, then we have $F_A = 0$ and $\psi = 0$.*

Proof. By the first of the YMD equations (1.2) and the Bianchi identity $d_A F_A = 0$, we have

$$\Delta_A F_A = d_A d_A^* F_A + d_A^* d_A F_A = d_A d_A^* F_A = d_A J(\psi) = Q_4(\nabla_A\psi, \psi), \tag{3.1}$$

where $\Delta_A = d_A d_A^* + d_A^* d_A$ and $Q_4(\nabla_A\psi,\psi)$ denotes a quadratic from on $\nabla_A\psi$ and ψ.

On the other hand, by the Weitzenböck formula (see [4,7]), we have

$$\Delta_A F_A = \nabla_A^*\nabla_A F_A + 4F_A + Q_5(F_A, F_A), \tag{3.2}$$

where $Q_5(F_A,F_A)$ is a quadratic form on F_A.

From (3.1) and (3.2) we obtain

$$\nabla_A^*\nabla_A F_A + 4F_A + Q_5(F_A,F_A) = Q_4(\nabla_A\psi,\psi). \tag{3.3}$$

Taking the L^2-inner product of (3.3) with F_A, we obtain by the Hölder's inequality

$$\begin{aligned}\|\nabla_A F_A\|_2^2 + 4\|F_A\|_2^2 &= -\big\langle Q_5(F_A,F_A), F_A\big\rangle + \big\langle Q_4(\nabla_A\psi,\psi), F_A\big\rangle \\ &\leqslant C\|F_A\|_2\|F_A\|_4^2 + C\|F_A\|_4\|\nabla_A\psi\|_{8/3}\|\psi\|_{8/3},\end{aligned} \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product.

On the other hand, by the second equation (1.3) of YMD and the Lichnerowicz–Weitzenböck formula (see [6,11]), we have

$$0 = \mathfrak{D}_A^2\psi = \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + c(F_A)\psi, \tag{3.5}$$

where $s = 4\times(4-1) = 12$ is the scalar curvature of S^4 and $c(F_A)$ is the Clifford multiplication by F_A, i.e., the composition

$$F_A \in C^\infty\big(\textstyle\bigwedge^2 T^*M \otimes \mathrm{End}(E)\big) \xrightarrow{\mathfrak{q}} C^\infty\big(Cl\big(T^*M\big)\otimes \mathrm{End}(E)\big) \xrightarrow{c} C^\infty\big(\mathrm{End}\big(\mathbb{S}(M)\big)\otimes E\big),$$

where $\mathrm{q}: \bigwedge^* T^*M \to Cl(T^*M)$ is the quantization map which is defined as the inverse of the symbol map $\sigma : Cl(T^*M) \to \bigwedge^* T^*M$ defined by $\sigma(e_{i_1} \cdot \cdots \cdot e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k}$ ($e_1, \ldots, e_m$ is an orthonormal frame of T^*M), see [6,11].

Taking the L^2-inner product of (3.5) with ψ, we obtain

$$\|\nabla_A \psi\|_2^2 + 3\|\psi\|_2^2 = -\langle c(F_A)\psi, \psi\rangle \leqslant C\|F_A\|_2\|\psi\|_4^2. \tag{3.6}$$

By (3.6), the Kato and the Sobolev inequalities, we obtain

$$\|\psi\|_{1,2}^2 := \|\nabla_A \psi\|_2^2 + \|\psi\|_2^2 \leqslant C\|F_A\|_2\|\psi\|_{1,2}^2. \tag{3.7}$$

From (3.7), if $\epsilon_0 > 0$ is small, we obtain $\psi \equiv 0$. Then by (3.4) we have

$$\|\nabla_A F_A\|_2^2 + 4\|F_A\|_2^2 \leqslant C\|F_A\|_2\|F_A\|_4^2. \tag{3.8}$$

From (3.8), the Kato and the Sobolev inequalities again we obtain

$$\|F_A\|_{1,2}^2 := \|\nabla_A F_A\|_2^2 + \|F_A\|_2^2 \leqslant C\|F_A\|_2\|F_A\|_{1,2}^2. \tag{3.9}$$

From this, if $\epsilon_0 > 0$ is small enough, we obtain $F_A = 0$. This completes the proof. $\square$

4. Energy quantization

In this section, we prove Theorem 1.2. We first prove Theorem 1.2(1).

Proof of Theorem 1.2(1). Let $\{(A_n, \psi_n)\}$ be as in the statement of Theorem 1.2. By the proof of Theorem 1.1, there exists $\epsilon_0 > 0$ such that if $E(A_n, \psi_n; B_r(a)) < \epsilon_0$, there exist a subsequence of $\{(A_n, \psi_n)\}$ (still denoted by $\{(A_n, \psi_n)\}$) and $g_n \in C^\infty(B_{r/2}(a), G)$ such that $\{(g_n^* A_n, g_n \cdot \psi_n)\}$ converges C^∞ in $B_{r/2}(a)$. Define $S = \bigcap_{r>0}\{a \in M\colon \liminf_{n\to\infty} \int_{B_r(a)} (|F_{A_n}|^2 + |\psi_n|^{8/3})\,\mathrm{dvol}_M \geqslant \epsilon_0\}$. Since $\sup_{n\geqslant 1} E(A_n, \psi_n) < \infty$, a simple covering argument shows that $\#S < \infty$ and by the above observation there exist gauge transformations g_n over $M \setminus S$ such that $\{(g_n^* A_n, g_n \cdot \psi_n)\}$ converges to some (A_0, ψ_0) in $C^\infty_{\mathrm{loc}}(M \setminus S)$. (A_0, ψ_0) is a solution to YMD in $M \setminus S$ and $E(A_0, \psi_0; M \setminus S) < \infty$. By the removable singularity theorem for solutions to YMD (see [16,13] and the remark at the end of the last section), (A_0, ψ_0) extends across the singularities S. This proves (1) of Theorem 1.2. $\square$

Set $S = \{p_1, \ldots, p_l\}$ ($l \geqslant 0$). If $l = 0$, there is nothing to prove. So we assume $l \geqslant 1$. Choose $r_i > 0$ ($i = 1, \ldots, l$) such that $B_{r_1}(p_i) \cap B_{r_j}(p_j) = \emptyset$ if $i \neq j$. In a suitable gauge, $\{(A_n, \psi_n)\}$ converges to (A_0, ψ_0) in $C^\infty(M \setminus \bigcup_{i=1}^l B_{r_i}(p_i))$. For simplicity of notation, we assume that $\{(A_n, \psi_n)\}$ itself converges to (A_0, ψ_0) in $C^\infty(M \setminus \bigcup_{i=1}^l B_{r_i}(p_i))$. The proof of Theorem 1.2(2) is completed if we can prove the following for each $1 \leqslant i \leqslant l$:

Claim 4.1. *Let p_i be as above. There exist $m \geqslant 1$ and solutions $(B_i, \varphi_1), \ldots, (B_m, \varphi_m)$ to the YMD on S^4 such that*

(A) $\lim_{\delta\downarrow 0} \lim_{n\to\infty} E_{\mathrm{YM}}(A_n; B_\delta(p_i)) = \sum_{j=1}^m E_{\mathrm{YM}}(B_j; S^4)$,
(B) $\lim_{\delta\downarrow 0} \lim_{n\to\infty} E_{\mathrm{Dirac}}(\psi_n; B_\delta(p_i)) = \sum_{j=1}^m E_{\mathrm{Dirac}}(\varphi_j; S^4)$.

For simplicity of notation, we drop the subscript of p_i and simply write it p.

Let $r = r_i > 0$ be chosen as before. Choose $\lambda_n > 0$ such that

$$\max_{x \in B_r(p)} E\big(A_n, \psi_n; B_{\lambda_n}(x)\big) = \epsilon_0/2 \tag{4.1}$$

and $x_n \in B_r(p)$ be a point at which the maximum of (4.1) is attained:

$$E\big(A_n, \psi_n; B_{\lambda_n}(x_n)\big) = \epsilon_0/2. \tag{4.2}$$

Then we have $\lambda_n \to 0$ and $x_n \to p$ as $n \to \infty$.

In normal coordinate on $B_r(p)$ at p, set $\tilde{A}_n(x) = \lambda_n A_n(x_n + \lambda_n x)$ and $\tilde{\psi}_n(x) = \lambda_n^{3/2}\psi_n(x_n + \lambda_n x)$. For any $R > 0$, $(\tilde{A}_n, \tilde{\psi}_n)$ is defined on $B_R(0)$ (with a metric $g_{\lambda_n, x_n} := \lambda_n^{-2}\rho^*_{\lambda_n, x_n} g$, where $\rho_{\lambda,x}(y) = x + \lambda y$) for large n and is a solution to the YMD. Notice that as $n \to \infty$, $g_{\lambda_n, x_n} \to g_0$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^4)$, where g_0 is the flat metric on $\mathbb{R}^4$.

By (4.1) and (4.2), for any $a \in \mathbb{R}^4$ we have

$$E\big(\tilde{A}_n, \tilde{\psi}_n; B_1(a)\big) = E\big(A_n, \psi_n; B_{\lambda_n}(\lambda_n a + x_n)\big) \leqslant \epsilon_0/2 < \epsilon_0 \tag{4.3}$$

for n large enough and

$$E\big(\tilde{A}_n, \tilde{\psi}_n; B_1(p)\big) = E\big(A_n, \psi_n; B_{\lambda_n}(x_n)\big) = \epsilon_0/2. \tag{4.4}$$

Moreover, for any $R > 0$, we have

$$E\big(\tilde{A}_n, \tilde{\psi}_n; B_R(p)\big) = E\big(A_n, \psi_n; B_{\lambda_n R}(x_n)\big) \leqslant C \tag{4.5}$$

for some $C > 0$ independent of large n.

From (4.3) and Corollary 2.1, there exist a subsequence of $\{(\tilde{A}_n, \tilde{\psi}_n)\}$ which we still denote by $\{(\tilde{A}_n, \tilde{\psi}_n)\}$ and (B_1, φ_1) a solution of YMD defined on $\mathbb{R}^4$ such that $(\tilde{A}_n, \tilde{\psi}_n)$ converges to (B_1, φ_1) in $C^\infty_{\mathrm{loc}}(\mathbb{R}^4)$ up to gauge transformations. Since we can take $R > 0$ arbitrary large in (4.5), we see that (B_1, φ_1) has a finite energy on $\mathbb{R}^4$. Since $\mathbb{R}^4$ is conformally equivalent to $S^4 \setminus \{\text{northpole}\}$, by the removable singularity theorem for YMD, (B_1, φ_1) extends to S^4 as a solution to YMD. We also call the extended solution (B_1, φ_1). This is the first bubble. It is non-trivial by (4.4).

We first prove equations (A) and (B) in Claim 4.1 under assuming $m = 1$, i.e., there are no other bubbles other than (B_1, φ_1). Since

$$\begin{aligned} E_{\mathrm{YM}}\big(A_n; B_\delta(x_n)\big) &= E_{\mathrm{YM}}\big(A_n; B_{\lambda_n R}(x_n)\big) + E_{\mathrm{YM}}\big(A_n; B_\delta(x_n) \setminus B_{\lambda_n R}(x_n)\big) \\ &= E_{\mathrm{YM}}\big(\tilde{A}_n; B_R(p)\big) + E_{\mathrm{YM}}\big(A_n; B_\delta(x_n) \setminus B_{\lambda_n R}(x_n)\big), \end{aligned} \tag{4.6}$$

it suffices to prove

$$\lim_{R\to\infty} \lim_{\delta\downarrow 0} \lim_{n\to\infty} E_{\mathrm{YM}}\big(A_n; B_\delta(x_n) \setminus B_{\lambda_n R}(x_n)\big) = 0. \tag{4.7}$$

For simplicity, we assume that the metric on $B_\delta(p)$ is flat. The general case is essentially the same since the metric g on $B_\delta(p)$ satisfies $g_{ij} = \delta_{ij} + O(|x|^2)$ in normal coordinate at p. With respect to the polar coordinate centered at x_n, we define $f_n : \mathbb{R} \times S^3 \to \mathbb{R}^4$ by $f_n(t, \theta) = (e^{-t}, \theta)$. We equip $\mathbb{R} \times S^3$ a metric $g = dt^2 + d\theta^2$ ($d\theta^2$ is the round metric on S^3) and $\mathbb{R}^4$ the flat metric. Then f_n is a conformal map with conformal factor e^{-t}. We then define $\hat{A}_n = f_n^* A_n$, $\hat{\psi}_n = e^{-3t/2} f_n^* \psi_n$. We have

$$E\big(\hat{A}_n, \hat{\psi}_n; \big[|\log\delta|, \infty\big) \times S^3\big) \leqslant C \tag{4.8}$$

for some $C > 0$ independent of n. Moreover, $(\hat{A}_n, \hat{\psi}_n)$ is a solution to YMD on $[|\log\delta|, \infty) \times S^3$.

Set $T_0 = |\log\delta|$ and $P_T = [T_0, T_0 + T] \times S^3$ for $T > 0$. We then have

$$(\hat{A}_n, \hat{\psi}_n) \to \big(f^* A_0, e^{-3t/2} f^* \psi_0\big) \quad \text{in } P_T \tag{4.9}$$

for any $T > 0$, where, with respect to the polar coordinate at p, $f : \mathbb{R} \times S^3 \to \mathbb{R}^4$ is defined by $f(t, \theta) = (e^{-t}, \theta)$.

From this we obtain

$$\lim_{n\to\infty} E(\hat{A}_n, \hat{\psi}_n; P_T) = E\big(A_0, \psi_0; B_\delta(p) \setminus B_{\delta e^{-T}}(p)\big). \tag{4.10}$$

On the other hand, since $E(A_0, \psi_0) < +\infty$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $E(A_0, \psi_0; B_\delta(p)) < \epsilon/2$. Thus for any $T > 0$, there exists $n(T) \geqslant 1$ such that for $n \geqslant n(T)$

$$E(\hat{A}_n, \hat{\psi}_n; P_T) < \epsilon/2. \tag{4.11}$$

Defining $T_n = |\log(\lambda_n R)|$ and $Q_{T,n} = [T_n - T, T_n] \times S^3$, since

$$\begin{aligned} E(\hat{A}_n, \hat{\psi}_n; Q_{T,n}) &= E\big(A_n, \psi_n; B_{\lambda_n R e^T}(x_n) \setminus B_{\lambda_n R}(x_n)\big) \\ &= E\big(\tilde{A}_n, \tilde{\psi}_n; B_{Re^T}(p) \setminus B_R(p)\big) \to E\big(B_1, \varphi_1; B_{Re^T}(p) \setminus B_R(p)\big) \end{aligned} \tag{4.12}$$

as $n \to \infty$ and $E(B_1, \varphi_1; \mathbb{R}^4) < +\infty$, there exists $R > 0$ and $n(R) \geqslant 1$ such that for $n \geqslant n(R)$ we have

$$E(\hat{A}_n, \hat{\psi}_n; Q_{T,n}) < \epsilon/2. \tag{4.13}$$

To proceed, we assert:

Claim 4.2. *Assume as above that there are no bubbles other than (B_1, φ_1). Then for any $\epsilon > 0$, there exists $N \geqslant 1$ such that for $n \geqslant N$, we have*

$$\int_{[t,t+1]\times S^3} |F_{\hat{A}_n}|^2 + |\hat{\psi}_n|^{\frac{8}{3}} \, \mathrm{dvol}_{[t,t+1]\times S^3} < \epsilon$$

for any $t \in [T_0, T_n - 1]$.

Proof. We prove the claim by contradiction. So suppose there exist $\epsilon > 0$ and $t_n \in [T_0, T_n - 1]$ such that

$$\int_{[t_n,t_n+1]\times S^3} |F_{\hat{A}_n}|^2 + |\hat{\psi}_n|^{\frac{8}{3}} \,\mathrm{dvol}_{[t_n,t_n+1]\times S^3} \geqslant \epsilon \tag{4.14}$$

for some subsequence (which we still denote by $\{n\}$).

By (4.11) and (4.13), we have $t_n - T_0 \to \infty$, $T_n - t_n \to \infty$ as $n \to \infty$. For simplicity of notation, we still denote by $(\hat{A}_n, \hat{\psi}_n)$ the t_n-translated fields $(\hat{A}_n(\cdot + t_n), \hat{\psi}_n(\cdot + t_n))$. Then by (4.14), we obtain

$$\int_{[0,1]\times S^3} |F_{\hat{A}_n}|^2 + |\hat{\psi}_n|^{\frac{8}{3}} \,\mathrm{dvol}_{[0,1]\times S^3} \geqslant \epsilon. \tag{4.15}$$

By (4.8) and the Uhlenbeck's theorem [28,32], there exits a subsequence of $\{(\hat{A}_n, \hat{\psi}_n)\}$ (which we still denote by $\{(\hat{A}_n, \hat{\psi}_n)\}$), a finite set Σ and fields $(\hat{A}_\infty, \hat{\psi}_\infty)$ defined on $\mathbb{R} \times S^3$ such that $(\hat{A}_n, \hat{\psi}_n) \rightharpoonup (\hat{A}_\infty, \hat{\psi}_\infty)$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times S^3 \setminus \Sigma) \times L^{8/3}_{\mathrm{loc}}(\mathbb{R} \times S^3 \setminus \Sigma)$ in some suitable gauge. Since $E(\hat{A}_\infty, \hat{\psi}_\infty; \mathbb{R} \times S^3 \setminus \Sigma) < \infty$ and $\mathbb{R} \times S^3$ is conformally equivalent to $S^4 \setminus \{\text{north pole, south pole}\}$, by the removable singularity theorem of YMD, $(\hat{A}_\infty, \hat{\psi}_\infty)$ can be extended to S^4 as a solution to YMD (which we still denote by $(\hat{A}_\infty, \hat{\psi}_\infty)$). If this convergence is strong in $W^{1,2} \times L^{8/3}$ on $[0,1] \times S^3$, then by (4.15) $(\hat{A}_\infty, \hat{\psi}_\infty)$ is non-trivial and we obtain a second bubble, a contradiction to our assumption. Thus the convergence is not strong on $[0,1] \times S^3$. Then there also arises the second bubble, a contradiction. In any case, we derive a contradiction from (4.14) and the claim is proved. □

Going back to $\mathbb{R}^4$, we obtain from Claim 4.2 that

$$\int_{B_r(x_n)\setminus B_{re^{-1}}(x_n)} |F_{A_n}|^2 \,dx < \epsilon \tag{4.16}$$

for any $\lambda_n R \leqslant r \leqslant \delta$.

Then by applying the result of [28], see also [20, Lemma III.1], we see that there exists a trivialization of the bundle over $B_\delta(x_n) \setminus B_{\lambda_n R}(x_n)$ such that

$$\|A_n\|_{L^4(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} + \|\nabla A_n\|_{L^2(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} \leqslant C\|F_{A_n}\|_{L^2(B_\delta(x_n)\setminus B_{\lambda_n R/2}(x_n))}. \tag{4.17}$$

On the other hand, by Claim 4.2 we have from Corollary 2.1 that

$$\|F_{A_n}\|_{L^\infty([t,t+1]\times S^3)} \leqslant C\epsilon^{1/2} \tag{4.18}$$

for $t \in [T_0, T_n - 1]$.

From (4.18), going back to $\mathbb{R}^4$ we obtain

$$|F_{A_n}|(x) \leqslant C\frac{\epsilon^{1/2}}{r^2} \tag{4.19}$$

for $x \in B_{\delta/2}(x_n) \setminus B_{2\lambda_n R}(x_n)$.

From (4.19), we have

$$\|F_{A_n}\|_{L^{2,\infty}(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))} \leqslant C\epsilon^{1/2}. \tag{4.20}$$

In (4.20), $L^{p,q}$ denotes the Lorentz space which is defined as the set of functions f such that $\|f\|_{L^{p,q}} := (\int_0^\infty [t^{1/p} f^*(t)]^q \times \frac{dt}{t})^{1/q} < \infty$ when $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $\|f\|_{L^{p,q}} = \sup_{t>0} t^{1/p} f^*(t) < \infty$ when $1 \leqslant p \leqslant \infty$, $q = \infty$, where f^* denotes the non-increasing rearrangement of f which is defined as $f^*(t) = \inf\{s\colon \lambda_f(s) \leqslant t\}$, where $\lambda_f(s) = \mathrm{meas}(\{x\colon |f(x)| > s\})$ (see [24] for more details).

In order to estimate $\|F_{A_n}\|_{L^2(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))}$, using the duality between $L^{2,\infty}$ and $L^{2,1}$, it is sufficient to estimate $\|F_{A_n}\|_{L^{2,1}(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))}$. To this end, we shall use YMD equations. By the Bianchi identity and the first of the YMD (1.2), we obtain

$$dF_{A_n} = -[A_n, F_{A_n}] \tag{4.21}$$

and

$$d^*F_{A_n} = -*[A_n, *F_{A_n}] + J(\psi_n). \tag{4.22}$$

By (4.21), we have

$$\|dF_{A_n}\|_{L^{4/3,1}} \leqslant \big\|[A_n, F_{A_n}]\big\|_{L^{4/3,1}} \leqslant C\|A_n\|_{L^{4,2}}\|F_{A_n}\|_{L^2}, \tag{4.23}$$

where all the norms in the above inequalities are taken on $B_{\delta/2}(x_n)$.

Let $\rho \in C^\infty(M)$ be a cutoff function such that $\rho = 1$ in $B_{\delta/3}(x_n)$ and $\mathrm{supp}(\rho) \subset B_{\delta/2}(x_n)$. By (2.5), we have

$$\mathfrak{D}_0(\rho\psi_n) = -\rho\sum_{\alpha=1}^{4}\sigma_*(A_{n,\alpha})e_\alpha\cdot\psi_n + \nabla\rho\cdot\psi_n. \tag{4.24}$$

By the Peetre–Sobolev embedding [19,25,26] we have $A_n \in W^{1,2}(B_{\delta/2}(x_n)) \subset L^{4,2}(B_{\delta/2}(x_n))$. Since $\psi_n \in L^{8/3} = L^{8/3,8/3}$, we obtain

$$\left\|\rho\sum_{\alpha=1}^{4}\sigma_*(A_{n,\alpha})e_\alpha\cdot\psi_n\right\|_{L^{8/5,8/7}(B_{\delta/2}(x_n))} \leqslant C\|A_n\|_{L^{4,2}(B_{\delta/2}(x_n))}\|\psi_n\|_{8/3}. \tag{4.25}$$

On the other hand, since $L^{8/3} \subset L^{8/5,8/7}$, we have

$$\|\nabla\rho\cdot\psi_n\|_{L^{8/5,8/7}(B_{\delta/2}(x_n))} \leqslant C\|\psi_n\|_{8/3}. \tag{4.26}$$

We regard $\rho\psi_n$ as a function on $\mathbb{R}^4$ (extended to the outside of $B_{\delta/2}(x_n)$ as 0). Denoting by Δ the Laplacian acting on functions on $\mathbb{R}^4$ rapidly decreasing at infinity, we have $\mathfrak{D}_0^2(\rho\psi_n) = \Delta(\rho\psi_n)$ and by (4.24)

$$\rho\psi_n = \Delta^{-1}\left(\mathfrak{D}_0\left(-\rho\sum_{\alpha=1}^{4}\sigma_*(A_{n,\alpha})e_\alpha\cdot\psi_n + \nabla\rho\cdot\psi_n\right)\right) = \Delta^{-1/2}\Delta^{-1/2}\mathfrak{D}_0\varphi_n, \tag{4.27}$$

where $\varphi_n = -\rho\sum_{\alpha=1}^{4}\sigma_*(A_{n,\alpha})e_\alpha\cdot\psi_n + \nabla\rho\cdot\psi_n$.

Since $\Delta^{-1/2}\mathfrak{D}_0 = \sum_{\alpha=1}^{4} e_\alpha\cdot\Delta^{-1/2}\nabla_{e_\alpha} = \sum_{\alpha=1}^{4} c(e_\alpha)R_\alpha$, where $R_\alpha = \Delta^{-1/2}\nabla_{e_\alpha}$ is the Riesz operator, we have from (4.27)

$$\rho\psi_n = \sum_{\alpha=1}^{4}\Delta^{-1/2}c(e_\alpha)R_\alpha\varphi_n. \tag{4.28}$$

Since R_α acts as a bounded operator on the Lorentz spaces (see [24]), we obtain

$$\left\|\sum_{\alpha=1}^{4}c(e_\alpha)R_\alpha\varphi_n\right\|_{L^{8/5,8/7}} \leqslant C\|\varphi_n\|_{L^{8/5,8/7}}. \tag{4.29}$$

Therefore we have

$$\big\|\Delta^{1/2}(\rho\psi_n)\big\|_{L^{8/5,8/7}} \leqslant C\|\varphi_n\|_{L^{8/5,8/7}}. \tag{4.30}$$

On the other hand, from $\nabla_\alpha = \Delta^{1/2}R_\alpha = R_\alpha\Delta^{1/2}$ and the boundedness of R_α in $L^{8/5,8/7}$, we obtain from (4.25), (4.26) and (4.30) that

$$\begin{aligned}\big\|\nabla_\alpha(\rho\psi_n)\big\|_{L^{8/5,8/7}} &= \big\|R_\alpha\Delta^{1/2}(\rho\psi_n)\big\|_{L^{8/5,8/7}}\\ &\leqslant C\big\|\Delta^{1/2}(\rho\psi_n)\big\|_{L^{8/5,8/7}}\\ &\leqslant C\|A_n\|_{L^{4,2}(B_{\delta/2}(x_n))}\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))}.\end{aligned} \tag{4.31}$$

Therefore we have

$$\|\nabla\psi_n\|_{L^{8/5,8/7}(B_{\delta/3}(x_n))} \leqslant C\|A_n\|_{L^{4,2}(B_{\delta/2}(x_n))}\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))}. \tag{4.32}$$

By the Peetre–Sobolev inequality ($\nabla\psi_n \in L^{8/5,8/7} \Rightarrow \psi_n \in L^{8/3,8/7}$ and $W^{1,2} \subset L^{4,2}$) and (4.32), we obtain

$$\begin{aligned}\|\psi_n\|_{L^{8/3,8/7}(B_{\delta/3}(x_n))} &\leqslant C\|A_n\|_{L^{4,2}(B_{\delta/2}(x_n))}\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))}\\ &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))},\end{aligned} \tag{4.33}$$

where $\|A\|_{W^{1,2}} = \|\nabla A\|_2 + \|A\|_4$.

Since $L^{8/3,8/7} \subset L^{8/3,2}$, by (4.33) we have

$$\|\psi_n\|_{L^{8/3,2}(B_{\delta/3}(x_n))} \leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n))}. \tag{4.34}$$

On the other hand, since $|J(\psi_n)| \leqslant C|\psi_n|^2$ we have from (4.34) that

$$\begin{aligned}\big\|J(\psi_n)\big\|_{L^{4/3,1}(B_{\delta/3}(x_n))} &\leqslant C\big\||\psi_n|^2\big\|_{L^{4/3,1}(B_{\delta/3}(x_n))}\\ &\leqslant C\|\psi_n\|^2_{L^{8/3,2}(B_{\delta/3}(x_n))}\\ &\leqslant C\|A_n\|^2_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}.\end{aligned}\tag{4.35}$$

Therefore by (4.22) and (4.35) we obtain:

$$\begin{aligned}\big\|d^*F_{A_n}\big\|_{L^{4/3,1}(B_{\delta/3}(x_n))} &\leqslant C\|A_n\|_{L^{4,2}(B_{\delta/2}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))} + C\big\|J(\psi_n)\big\|_{L^{4/3,1}(B_{\delta/3}(x_n))}\\ &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))} + C\|A_n\|^2_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}\\ &\quad + C\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}.\end{aligned}\tag{4.36}$$

As in [20], define G as the solution of $dG = dF_{A_n}$, $d^*G = d^*F_{A_n}$ in $B_{\delta/3}(0)$ and $\iota^*_{\partial B_{\delta/3}(x_n)}G = 0$, where $\iota_{B_{\delta/3}(x_n)} : \partial B_{\delta/3}(x_n) \to B_{\delta/3}(x_n)$ is the inclusion. By (4.23), (4.36) and the boundedness of the Riesz operator in the Lorentz spaces again, we have

$$\begin{aligned}\|G\|_{L^2(B_{\delta/3}(x_n))} + \|\nabla G\|_{L^{4/3,1}(B_{\delta/3}(x_n))} &\leqslant C\|dG\|_{L^{4/3,1}(B_{\delta/3}(x_n))} + \big\|d^*G\big\|_{L^{4/3,1}(B_{\delta/3}(x_n))}\\ &\leqslant C\|dF_{A_n}\|_{L^{4/3,1}(B_{\delta/3}(x_n))} + C\big\|d^*F_{A_n}\big\|_{L^{4/3,1}(B_{\delta/3}(x_n))}\\ &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))}\\ &\quad + C\|A_n\|^2_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))} + C\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}.\end{aligned}\tag{4.37}$$

Since $F_{A_n} - G$ is harmonic, we have by (4.37)

$$\begin{aligned}\big\|\nabla(F_{A_n} - G)\big\|_{L^{4/3,1}(B_{\delta/4}(x_n))} &\leqslant C\|F_{A_n}\|_{L^2(B_{\delta/3}(x_n))} + C\|G\|_{L^2(B_{\delta/3}(x_n))}\\ &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))} + C\|A_n\|^2_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}\\ &\quad + C\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))} + C\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))}.\end{aligned}\tag{4.38}$$

By (4.37) and (4.38), we obtain

$$\begin{aligned}\|\nabla F_{A_n}\|_{L^{4/3,1}(B_{\delta/4}(x_n))} &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))} + C\|A_n\|^2_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}\\ &\quad + C\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))} + C\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))}.\end{aligned}\tag{4.39}$$

By the Peetre–Sobolev embedding ($\nabla F_A \in L^{4/3,1} \Rightarrow F_A \in L^{2,1}$) and (4.39), we have

$$\begin{aligned}\|F_{A_n}\|_{L^{2,1}(B_{\delta/4}(x_n))} &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))} + C\|A_n\|^2_{W^{1,2}(B_{\delta/2}(x_n))}\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))}\\ &\quad + C\|\psi_n\|^2_{L^{8/3}(B_{\delta/2}(x_n))} + C\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n))}.\end{aligned}\tag{4.40}$$

Since the bound of the norm $\|A_n\|_{W^{1,2}} = \|\nabla A_n\|_{L^2} + \|A_n\|_{L^4}$ is given only on $B_{\delta/2}(x_n) \setminus B_{\lambda_n R}(x_n)$ (see (4.17)), we need to estimate F_{A_n} on this set.

For this, let η be a cutoff function such that $\eta = 1$ in $B_{\delta/2}(x_n) \setminus B_{2\lambda_n R}(x_n)$, $\eta = 0$ in $B_{\lambda_n R}(x_n)$ and $\|\nabla\eta\|_\infty \leqslant C(R\lambda_n)^{-1}$. First we have from (4.21) and (4.22) that

$$d\big(\eta^2 F_{A_n}\big) = \eta^2\, dF_{A_n} + 2\eta\, d\eta \wedge F_{A_n} = -\eta^2[A_n, F_{A_n}] + 2\eta\, d\eta \wedge F_{A_n},\tag{4.41}$$

$$d\big(*\big(\eta^2 F_{A_n}\big)\big) = \eta^2 d(*F_{A_n}) + 2\eta\, d\eta \wedge *F_{A_n} = -\eta^2[A_n, *F_{A_n}] + 2\eta\, d\eta \wedge *F_{A_n} + *\big(\eta^2 J(\psi_n)\big),\tag{4.42}$$

$$\big\|\eta^2[A_n, F_{A_n}]\big\|_{L^{4/3,1}(B_{\delta/2}(x_n))} \leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))}\|F_{A_n}\|_{L^2(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} \leqslant C\tag{4.43}$$

and similarly

$$\big\|\eta^2[A_n, *F_{A_n}]\big\|_{L^{4/3,1}(B_{\delta/2}(x_n)} \leqslant C.\tag{4.44}$$

We next estimate $\|\eta\, d\eta \wedge F_{A_n}\|_{L^{4/3,1}(B_{\delta/2}(x_n))}$ and $\|\eta\, d\eta \wedge *F_{A_n}\|_{L^{4/3,1}(B_{\delta/2}(x_n))}$.

On $\mathrm{supp}(d\eta)$, we have $r \sim R\lambda_n$ and by (4.19) we obtain

$$|\eta\, d\eta \wedge F_{A_n}| \leqslant C(R\lambda_n)^{-1}\frac{\sqrt{\epsilon}}{r^2} \leqslant C\frac{\sqrt{\epsilon}}{r^3}\tag{4.45}$$

in $B_{\delta/2}(x_n)$.

Writing $f = |\eta\, d\eta \wedge F_{A_n}|$, by (4.45) we easily deduce that $\lambda_f(s) = \mathrm{meas}(\{x \in B_{\delta/2}(x_n)\colon\ |f(x)| > s\})$ satisfies $\lambda_f(s) = 0$ if $s > C\frac{\sqrt{\epsilon}}{(\lambda_n R)^3}$ and $\lambda_f(s) \leqslant C(\lambda_n R)^4$ if $0 \leqslant s \leqslant C\frac{\sqrt{\epsilon}}{(\lambda_n R)^3}$ for some constant $C > 0$ independent of n, R, ϵ, etc. From this, we have that $f^*(t) = \inf\{s > 0\colon\ \lambda_f(s) \leqslant t\}$ satisfies $f^*(t) = 0$ for $t \geqslant C'(\lambda_n R)^4$ and $f^*(t) \leqslant C\frac{\sqrt{\epsilon}}{(\lambda_n R)^3}$ for $0 \leqslant t < C'(\lambda_n R)^4$ for some another constant $C' > 0$ independent of n, R, ϵ, etc. Thus we have

$$\|f\|_{L^{4/3,1}} = \int_0^\infty t^{3/4} f^*(t)\frac{1}{t}\,dt \leqslant \int_0^{C'(\lambda_n R)^4} t^{-1/4} C' \frac{\sqrt{\epsilon}}{(\lambda_n R)^3}\,dt = C\sqrt{\epsilon}. \tag{4.46}$$

Therefore we obtain

$$\|\eta\, d\eta \wedge F_{A_n}\|_{L^{4/3,1}(B_{\delta/2}(x_n))} \leqslant C\sqrt{\epsilon}. \tag{4.47}$$

Similarly, we have

$$\|\eta\, d\eta \wedge *F_{A_n}\|_{L^{4/3,1}(B_{\delta/2}(x_n))} \leqslant C\sqrt{\epsilon}. \tag{4.48}$$

We next estimate $\|\eta^2 J(\psi_n)\|_{L^{4/3,1}(B_{\delta/2}(x_n))}$. By (2.5), we have

$$\mathfrak{D}_0(\eta\psi_n) = -\eta \sum_{\alpha=1}^{4} \sigma_*(A_{n,\alpha}) e_\alpha \cdot \psi_n + \nabla\eta \cdot \psi_n. \tag{4.49}$$

On the other hand, we have by the Peetre–Sobolev inequality

$$\begin{aligned}\left\|\eta \sum_{\alpha=1}^{4} \sigma_*(A_{n,\alpha}) e_\alpha \cdot \psi_n\right\|_{L^{8/5,8/7}(B_{\delta/2}(x_n))} &\leqslant C\|A_n\|_{L^{4,2}(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} \|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} \\ &\leqslant C\|A_n\|_{W^{1,2}(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} \|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))}.\end{aligned} \tag{4.50}$$

By Corollary 2.1 and Claim 4.2, we have $\|\hat{\psi}_n\|_{L^\infty([t,t+1]\times S^3)} \leqslant C\epsilon^{3/8}$. Going back to $\mathbb{R}^4$, we therefore have

$$\left|\psi_n(x)\right| \leqslant C\epsilon^{3/8} r^{-3/2} \tag{4.51}$$

for $2\lambda_n R \leqslant r \leqslant \delta/2$.

Setting $g = |\nabla\eta \cdot \psi_n|$, we have $g(x) \leqslant C\epsilon^{3/8}(\lambda_n R)^{-5/2}$ on $\lambda_n R \leqslant r \leqslant 2\lambda_n R$ and $g = 0$ otherwise. Then we have $\lambda_g(s) = 0$ if $s > C\epsilon^{3/8}(\lambda_n R)^{-5/2}$ and $\lambda_g(s) \leqslant C'(\lambda_n R)^4$ for $0 \leqslant s \leqslant C\epsilon^{3/8}(\lambda_n R)^{-5/2}$, where C, C' are as before. Then we have $g^*(t) = 0$ if $t \geqslant C(\lambda_n R)^4$ and $g^*(t) \leqslant C\epsilon^{3/8}(\lambda_n R)^{-5/2}$ if $0 \leqslant t \leqslant C(\lambda_n R)^4$. Thus we have

$$\begin{aligned}\|g\|_{L^{8/5,8/7}(B_{\delta/2}(x_n))} &= \left(\int_0^\infty \left(t^{5/8} g^*(t)\right)^{8/7} \frac{dt}{t}\right)^{7/8} = \left(\int_0^\infty t^{-2/7} g^*(t)^{8/7}\,dt\right)^{7/8} \\ &\leqslant \left(\int_0^{C'(\lambda_n R)^4} t^{-2/7}\left(C\epsilon^{3/8}(\lambda_n R)^{-5/2}\right)^{8/7} dt\right)^{7/8} \leqslant C\epsilon^{3/8}.\end{aligned} \tag{4.52}$$

Therefore we obtain

$$\|\nabla\eta \cdot \psi_n\|_{L^{8/5,8/7}(B_{\delta/2}(x_n))} \leqslant C\epsilon^{3/8}. \tag{4.53}$$

Once (4.50) and (4.53) are obtained, we obtain as before that $\|\nabla(\eta\psi_n)\|_{L^{8/5,8/7}(B_{\delta/2}(x_n))}$ is bounded and therefore $\|\eta\psi_n\|_{L^{8/3,8/7}(B_{\delta/2}(x_n))}$ is also bounded. This in particular implies that $\|\eta\psi_n\|_{L^{8/3,2}(B_{\delta/2}(x_n))}$ is bounded and we obtain

$$\left\|\eta^2 J(\psi_n)\right\|_{L^{4/3,1}(B_{\delta/2}(x_n))} \leqslant C\left\|\eta^2|\psi_n|^2\right\|_{L^{4/3,1}(B_{\delta/2}(x_n))} \leqslant C\|\eta\psi_n\|^2_{L^{8/3,2}(B_{\delta/2}(x_n))} \leqslant C \tag{4.54}$$

independent of n.

From (4.41), (4.42), (4.43), (4.44), (4.47), (4.48) and (4.54), we obtain

$$\left\|d\left(\eta^2 F_{A_n}\right)\right\|_{L^{4/3,1}(B_{\delta/2}(x_n))}, \qquad \left\|d * \left(\eta^2 F_{A_n}\right)\right\|_{L^{4/3,1}(B_{\delta/2}(x_n))} \leqslant C \tag{4.55}$$

for all n.

From (4.55), as before, we obtain the bound

$$\|F_{A_n}\|_{L^{2,1}(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))} \leqslant C \tag{4.56}$$

independent of n.

By (4.20) and (4.56), we therefore obtain

$$\begin{aligned}\|F_{A_n}\|^2_{L^2(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))} &\leqslant \|F_{A_n}\|_{L^{2,1}(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))}\|F_{A_n}\|_{L^{2,\infty}(B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))}\\ &\leqslant C\epsilon^{1/2}.\end{aligned} \tag{4.57}$$

Since $\epsilon > 0$ is arbitrary, we finally obtain

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E_{\mathrm{YM}}\big(A_n; B_\delta(x_n)\setminus B_{\lambda_n R}(x_n)\big) = 0. \tag{4.58}$$

This completes the proof of (A) under assuming $m = 1$.

We next estimate $E_{\mathrm{Dirac}}(\psi_n; B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n))$. Take a cutoff function $\rho \in C_0^\infty(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))$ such that $\rho = 1$ in $B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n)$, $|\nabla\rho| \leqslant C\delta^{-1}$ in $B_{\delta/2}(x_n)\setminus B_{\delta/4}(x_n)$ and $|\nabla\rho| \leqslant C(\lambda_n R)^{-1}$ in $B_{2\lambda_n R}(x_n)\setminus B_{\lambda_n R}(x_n)$, where $C > 0$ is a constant independent of n and δ.

By (4.24), elliptic estimate for $\mathfrak{D}_0$ and the Sobolev embedding $W^{1,8/5} \subset L^{8/3}$, we have

$$\begin{aligned}\|\rho\psi_n\|_{L^{8/3}} &\leqslant C\big\|\mathfrak{D}_0(\rho\psi_n)\big\|_{L^{8/5}}\\ &\leqslant C\left\|-\rho\sum_{\alpha=1}^{4}\sigma_*(A_{n,\alpha})e_\alpha\cdot\psi_n + \nabla\rho\cdot\psi_n\right\|_{L^{8/5}}\\ &\leqslant C\|A_n\|_{L^4(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))}\|\psi_n\|_{L^{8/3}(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} + C\|\nabla\rho\cdot\psi_n\|_{L^{8/5}}.\end{aligned} \tag{4.59}$$

Here, we have by the Hölder's inequality and Claim 4.2 that

$$\begin{aligned}\|\nabla\rho\cdot\psi_n\|^{8/5}_{L^{8/5}} &= \int\limits_{B_{\delta/2}(x_n)\setminus B_{\delta/4}(x_n)} |\nabla\rho\cdot\psi_n|^{8/5}\,dx + \int\limits_{B_{2\lambda_n R}(x_n)\setminus B_{\lambda_n R}(x_n)} |\nabla\rho\cdot\psi_n|^{8/5}\,dx\\ &\leqslant C\delta^{-8/5}\int\limits_{B_{\delta/2}(x_n)\setminus B_{\delta/4}(x_n)} |\psi_n|^{8/5}\,dx + C(\lambda_n R)^{-8/5}\int\limits_{B_{2\lambda_n R}(x_n)\setminus B_{\lambda_n R}(x_n)} |\psi_n|^{8/5}\,dx\\ &\leqslant C\Bigg(\int\limits_{B_{\delta/2}(x_n)\setminus B_{\delta/4}(x_n)} |\psi_n|^{8/3}\,dx\Bigg)^{3/5} + C\Bigg(\int\limits_{B_{2\lambda_n R}(x_n)\setminus B_{\lambda_n R}(x_n)} |\psi_n|^{8/3}\,dx\Bigg)^{3/5}\\ &\leqslant C\epsilon^{3/5}.\end{aligned} \tag{4.60}$$

By (4.60) and $\|A_n\|_{L^4(B_{\delta/2}(x_n)\setminus B_{\lambda_n R}(x_n))} \leqslant C\|F_{A_n}\|_{L^2(B_\delta(x_n)\setminus B_{\lambda_n R/2}(x_n))}$ (see (4.17)), we obtain

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E_{\mathrm{Dirac}}\big(\psi_n; B_{\delta/4}(x_n)\setminus B_{2\lambda_n R}(x_n)\big) \leqslant C\epsilon. \tag{4.61}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof of (B) for under assuming $m = 1$.

The argument is similar for the case where we do not assume $m = 1$. We only sketch the proof for the case $m \leqslant 2$, i.e., there is at most one bubble besides (B_1, φ_1). If the assertion of Claim 4.2 holds, we are done since, as we have seen, there is only one bubble (B_1, φ_1) in this case. Thus we assume there are $\epsilon > 0$ and $t_n \in [T_0, T_n - 1]$ such that

$$\int\limits_{[t_n,t_n+1]\times S^3} |F_{\hat{A}_n}|^2 + |\hat{\psi}_n|^{8/3}\,\mathrm{dvol}_{[t_n,t_n+1]\times S^3} \geqslant \epsilon \tag{4.62}$$

and $t_n - T_0,\ T_n - t_n \to \infty$ $(n \to \infty)$.

For simplicity, we use the same notation $(\hat{A}_n, \hat{\psi}_n)$ for t_n-translated fields. We then have

$$\int\limits_{[0,1]\times S^3} |F_{\hat{A}_n}|^2 + |\hat{\psi}_n|^{8/3}\,\mathrm{dvol}_{[0,1]\times S^3} \geqslant \epsilon \tag{4.63}$$

and $(\hat{A}_n, \hat{\psi}_n)$ is defined on $[T_0 - t_n, T_n - t_n] \times S^3 =: [-a_n, b_n] \times S^3$, where $a_n = t_n - T_0 \to \infty$ and $b_n = T_n - t_n \to \infty$ as $n \to \infty$.

There are two possibilities: (i) $(\hat{A}_n, \hat{\psi}_n)$ strongly converges to some $(\hat{A}_\infty, \hat{\psi}_\infty)$ in $[0,1] \times S^3$ or (ii) $(\hat{A}_n, \hat{\psi}_n)$ does not converge strongly in $[0,1] \times S^3$. For the first case, $(\hat{A}_\infty, \hat{\psi}_\infty)$ gives the second bubble which we denote by (B_2, φ_2) and for the second case, after suitable conformal transformations one also obtains the second bubble (B_2, φ_2). In either case, we need to show that $\lim_{n\to\infty} E(\hat{A}_n, \hat{\psi}_n; [-a_n, b_n] \times S^3) = E(B_2, \varphi_2; S^4)$. We first consider the case (i). Since $m \leqslant 2$, there are no energy concentrations and we have

$$(\hat{A}_n, \hat{\psi}_n) \to (\hat{A}_\infty, \hat{\psi}_\infty) \quad \text{in } W^{1,2}_{\mathrm{loc}}\big(\mathbb{R} \times S^3\big) \times L^{8/3}_{\mathrm{loc}}\big(\mathbb{R} \times S^3\big),$$

i.e.,

$$\lim_{M\to\infty} \lim_{n\to\infty} E\big(\hat{A}_n, \hat{\psi}_n; [-M, M] \times S^3\big) = E\big(B_2, \varphi_2; S^4\big). \tag{4.64}$$

On the other hand, similar to the case $m = 1$, we see that the energy of $(\hat{A}_n, \hat{\psi}_n)$ is small on $[-a_n, -M] \times S^3$ and $[M, b_n] \times S^3$ if $\delta > 0$ is small and n, R, M large enough. Therefore we have in this case that $\lim_{n\to\infty} E(\hat{A}_n, \hat{\psi}_n; [-a_n, b_n] \times S^3)$ is arbitrary close to $E(B_2, \varphi_2; S^4)$ if $\delta > 0$ is small and $R > 0$ is large enough.

For the case (ii), after a conformal transformation if necessary, we may assume that there is a bubble at $(0, N) \in [-1,1] \times S^3$ (N is the north pole of S^3). Since $m \leqslant 2$, there are no more bubbles and as in the case $m = 1$, we have

$$\lim_{n\to\infty} E\big(\hat{A}_n, \hat{\psi}_n; [-1, 1] \times S^3\big) = E\big(B_2, \varphi_2; S^4\big). \tag{4.65}$$

As in the case $m = 1$, the energies on $[-a_n, -1] \times S^3$ and $[1, b_n] \times S^3$ are arbitrary small if $\delta > 0$ small and n, R large. Thus in this case $\lim_{n\to\infty} E(\hat{A}_n, \hat{\psi}_n; [-1, 1] \times S^3)$ is arbitrary close to $E(B_2, \varphi_2; S^4)$ if $\delta > 0$ is small and R is large enough. Therefore the proof is completed under our assumption. By Proposition 3.1, we know that there are at most a finite number of bubbles and the proof for the general case is completed by the similar argument.

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